Heisenberg Uncertainty Relation and the quantum to classical transition

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John S. Briggs

Short History of the HUR

Born 1926

$$\hat{p}\,\hat{x} - \hat{x}\,\hat{p} = [\hat{p}, \hat{x}] = -i\hbar$$

Jordan 1926:

Fourier relation between x and p spaces

Heisenberg 1927

$$\Delta p\, \Delta x \sim \hbar$$

Robertson 1929

$$\Delta p \Delta x \geq \frac{1}{2} |\langle [\hat{p}, \hat{x}] \rangle| = \hbar/2$$

$$\Delta x \Delta p \geq \hbar/2$$

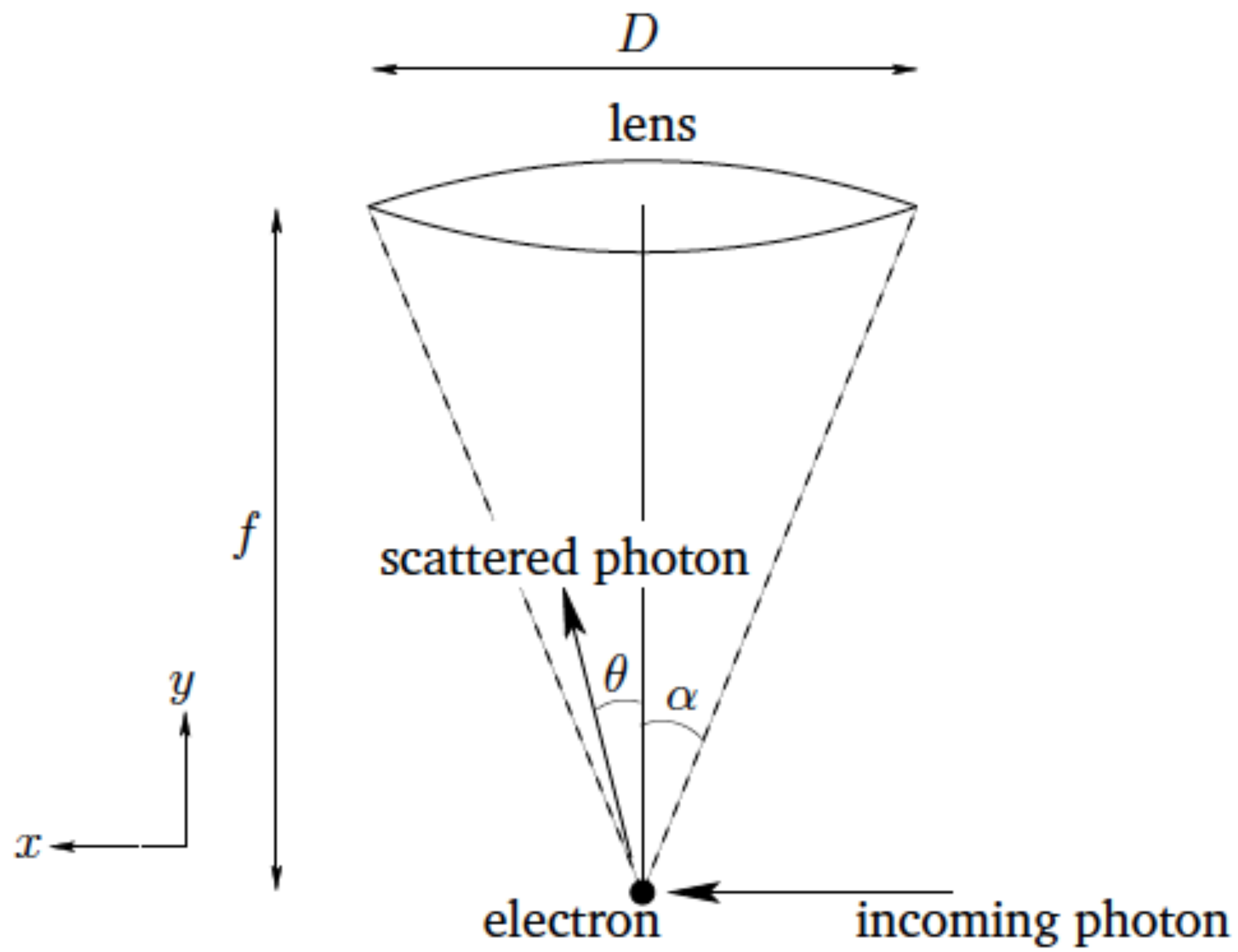


Figure 3.8: *Heisenberg's microscope.*

This is the famous *Heisenberg uncertainty principle*, first proposed by Werner Heisenberg in 1927. According to this principle, it is impossible to simultaneously measure the position and momentum of a particle (exactly). Indeed, a good knowledge of the particle's position implies a poor knowledge of its momentum, and *vice versa*. Note that the uncertainty principle is a direct consequence of representing particles as waves.

Robertson 1929

Physical Review 34, 136 (1929)

$$\Delta A \Delta B \geq \frac{1}{2}|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|$$

The $\Delta A, \Delta B$ *were identified with the uncertainties of Heisenberg's simultaneous measurement*

$$\Delta A = [\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2]^{1/2}$$

is a statistical property of an ensemble of identical systems

Uncertainty relation pairs

$$\Delta p, \Delta x$$

$$\Delta J, \Delta \phi$$

$$\Delta E, \Delta t$$ **Fourier transform, no time operator**

Imaging Theorem, Quantum to “classical” transition

$$\Psi(x,t) \rightarrow \Psi(x(t)) \approx \tilde{\Psi}(p,t)$$

The Generalised Imaging Theorem

$$|\Psi(t)\rangle = e^{\frac{i}{\hbar}H(t-t')}|\Psi(t')\rangle$$

$$\langle \mathbf{r}_f|\Psi(t_f)\rangle = \int \langle \mathbf{r}_f|e^{\frac{i}{\hbar}H(t_f-t_i)}|\mathbf{p}\rangle\langle\mathbf{p}|\Psi(t_i)\rangle \; d\mathbf{p}$$

$$\Psi(\mathbf{r}_f, t_f) = \int K(\mathbf{r}_f, t_f : \mathbf{p}, t_i)\tilde{\Psi}(\mathbf{p}, t_i) \; d\mathbf{p}$$

Asymptotically large time

The semi-classical approximation for the time-development operator

$$\langle \mathbf{r}_f|e^{\frac{i}{\hbar}H(t_f-t_i)}|\mathbf{p}\rangle \approx \left|\det\frac{\partial^2 \mathrm{S_c}}{\partial \mathbf{r}_\mathrm{f}\partial\mathbf{p}}\right|^{1/2} \exp\left(\frac{i}{\hbar}S_c(\mathbf{r}_f, t_f; \mathbf{p}, t_i)\right)$$

Integral in Stationary Phase Approximation

Free motion, one dimension

$$S_c = px - \frac{p^2}{2m}t \qquad t \equiv t_f - t_i$$

Stationary phase point

$$\frac{dS_c}{dp} = 0 \qquad x = pt/m, \qquad p = mx/t$$

$$\Psi(x,t) = \left(\frac{dp}{dx}\right)^{1/2} \exp\left(\frac{i}{\hbar}S_c(t)\right)\tilde{\Psi}(p,t)$$

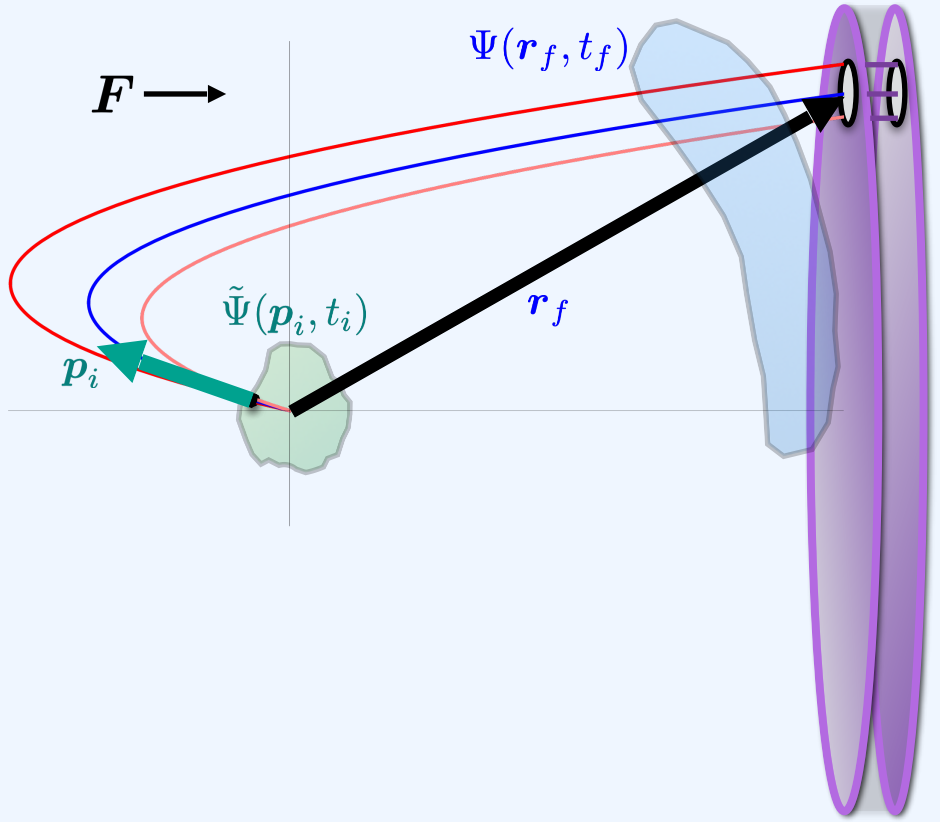
classical

classical

quantum

$$|\Psi(x,t)|^2 = \left(\frac{dp}{dx}\right) |\tilde{\Psi}(p,t)|^2$$

$$|\Psi(x,t)|^2 \, dx(t) = |\tilde{\Psi}(p,t)|^2 \, dp$$



Einstein 1927

Indeed, if the particle is spread out in space before being detected, the fact that it is always detected at a given point implies that it condenses itself on that point and that its presence vanishes elsewhere. Thus something nonlocal must be taking place.

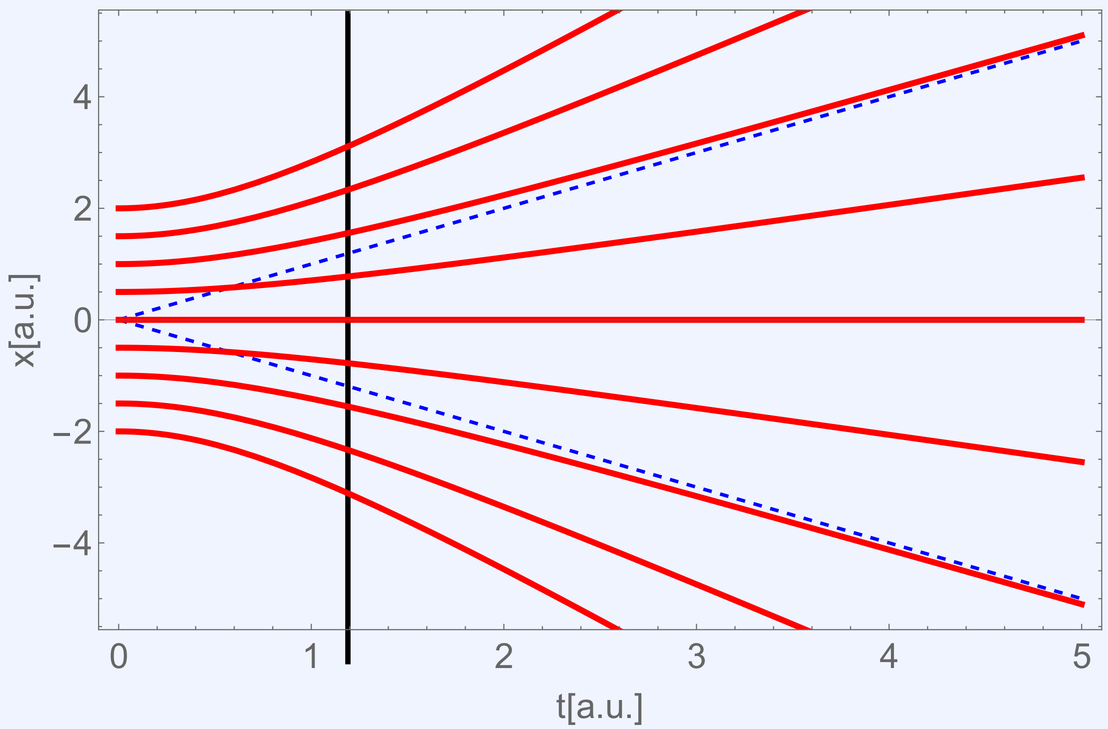
Einstein adds:

> In my opinion, one can remove this objection [action at a distance] only in the following way, that one does not describe the process solely by the Schrodinger wave, but that at the same time one localises the particle during the propagation.

ImagingTheorem !

Spreading Free Gaussian Wavepacket and the Quantum to Classical Transition

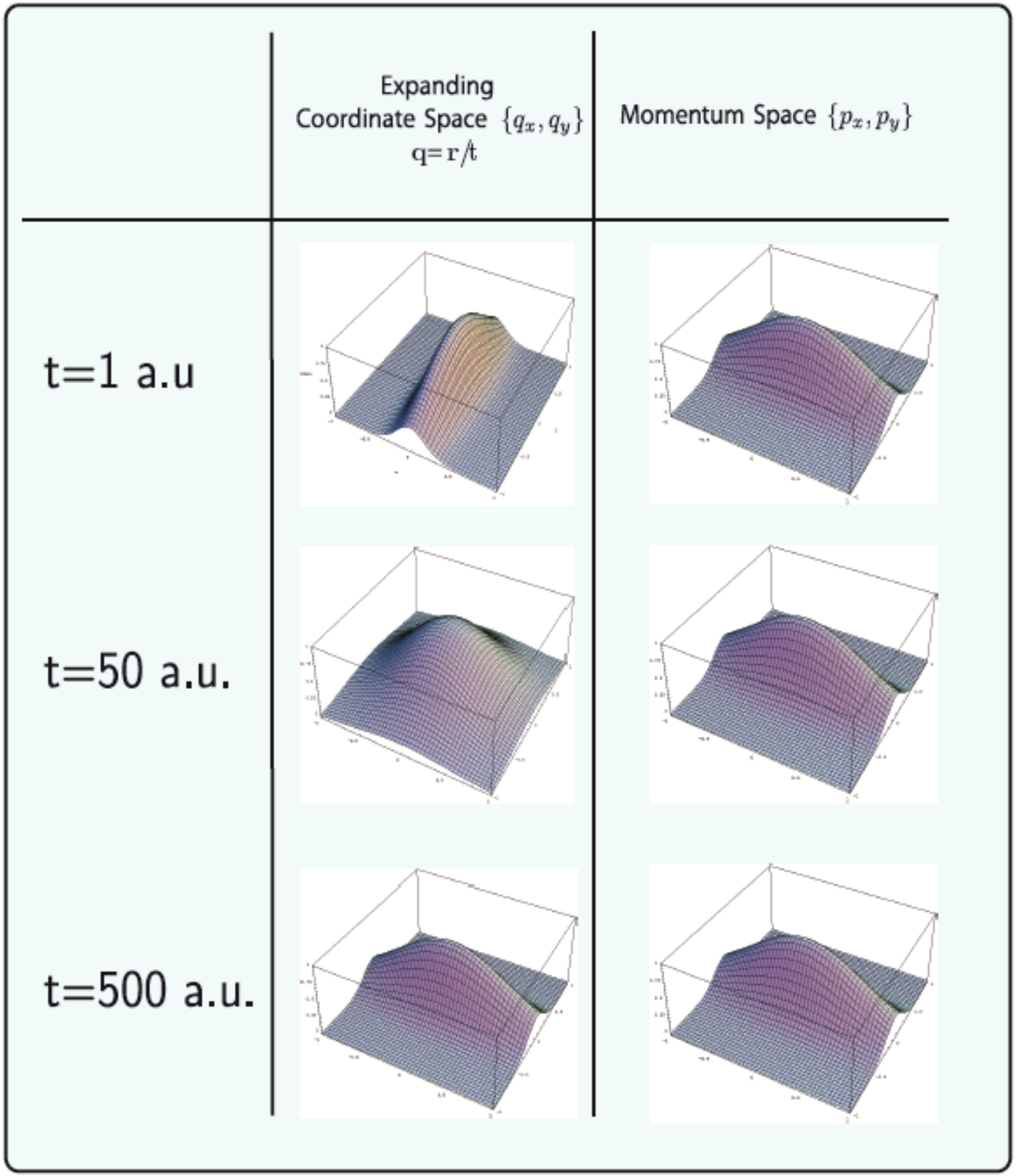
QM



T

$$T = \left(\frac{m}{\hbar}\right)\sigma^2$$

$$t \gg T \qquad x \approx vt$$



Macek J H 2012 *Dynamical Processes in Atomic and Molecular Physics* ed G Ogurtsov and D Dowek (Oak Park, IL: Bentham Science)

Propagation in Time and Space

$$\Psi(x,t) = \int K(x,t : x',t')\,\Psi(x',t')\,dx'$$

$$K(x,t : x',t') = \langle x,t|\exp\left(-\frac{i}{\hbar}H(t-t')\right)|x',t'\rangle$$

Free motion, t'= 0

$$K(x,t : x',t') = \left(\frac{m}{2\pi\hbar t}\right)^{1/2}\exp\left(i\frac{m}{2\hbar t}(x-x')^2\right)$$

$$\hat{p}\,K(x,t : x',0) = -i\hbar\frac{\partial K}{\partial x} = \frac{m(x-x')}{t}\,K(x,t : x',0)$$

$$\hat{p}\,\Psi(x,t) = \frac{mx}{t}\Psi(x,t) - \frac{m}{t}\int x'\,K(x,t:x',0)\,\Psi(x',0)\,dx'$$

At infinitely large time

$$\hat{p}\,\Psi(x,t) = \frac{mx}{t}\Psi(x,t) = p\,\Psi(x,t)$$

The space wave function is an eigenfunction of the momentum operator.

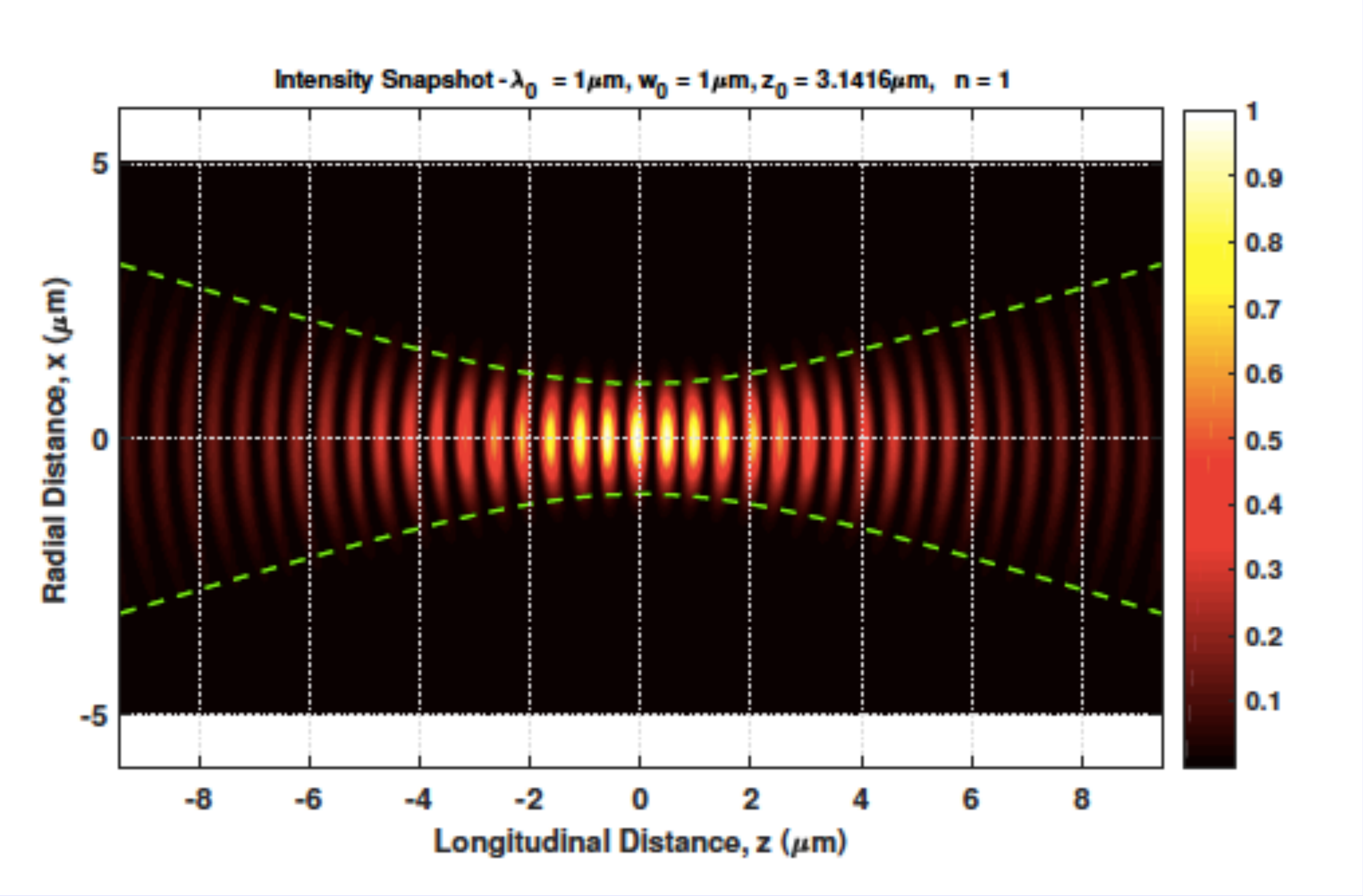
At finitely large time

$$\Delta x\,\Delta p \approx \Delta x\,\left(\frac{m\Delta x}{t}\right) = \frac{m}{t}(\Delta x)^2$$

can be made much smaller than $\hbar$

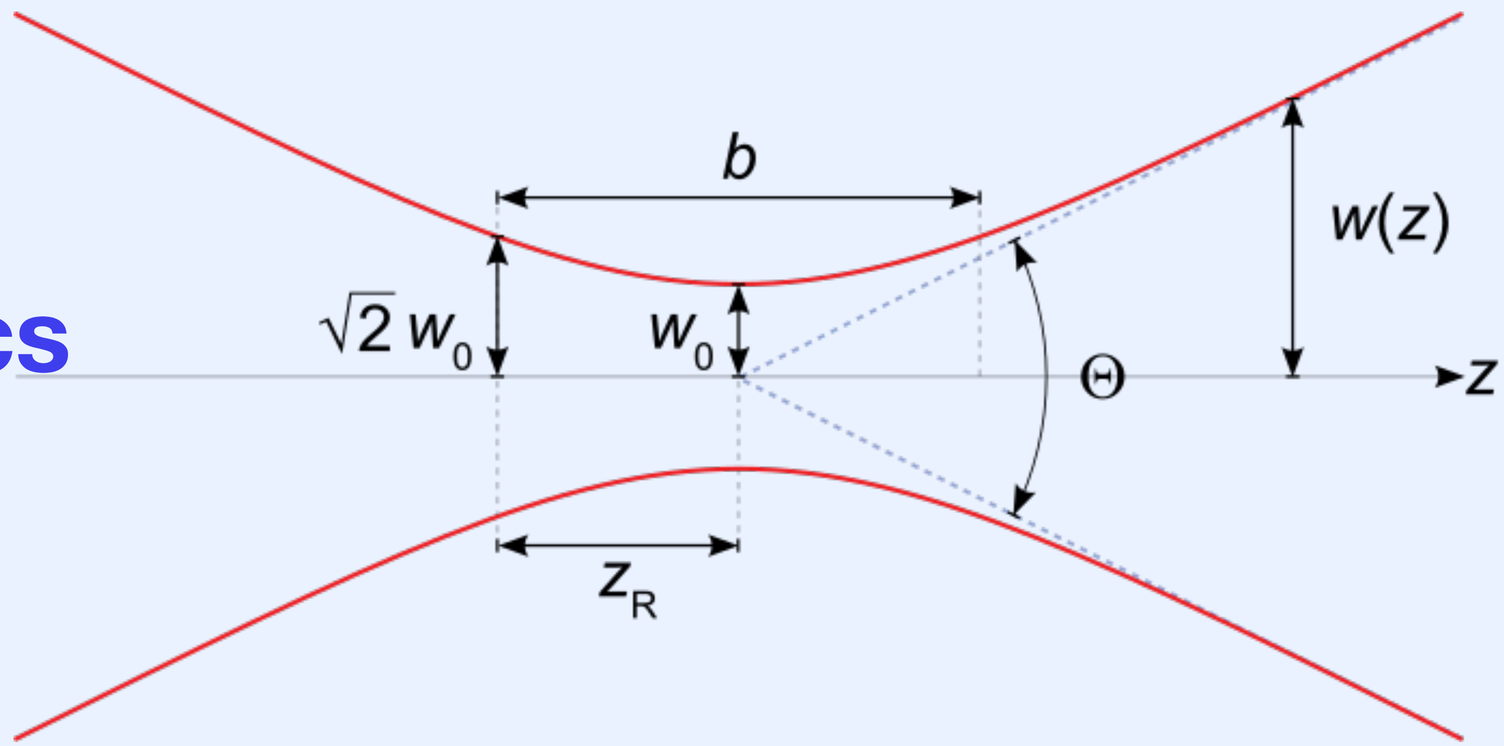
Expanding wave packet
and the quantum to classical transition

Expanding wave packet
and the wave to beam optics transition



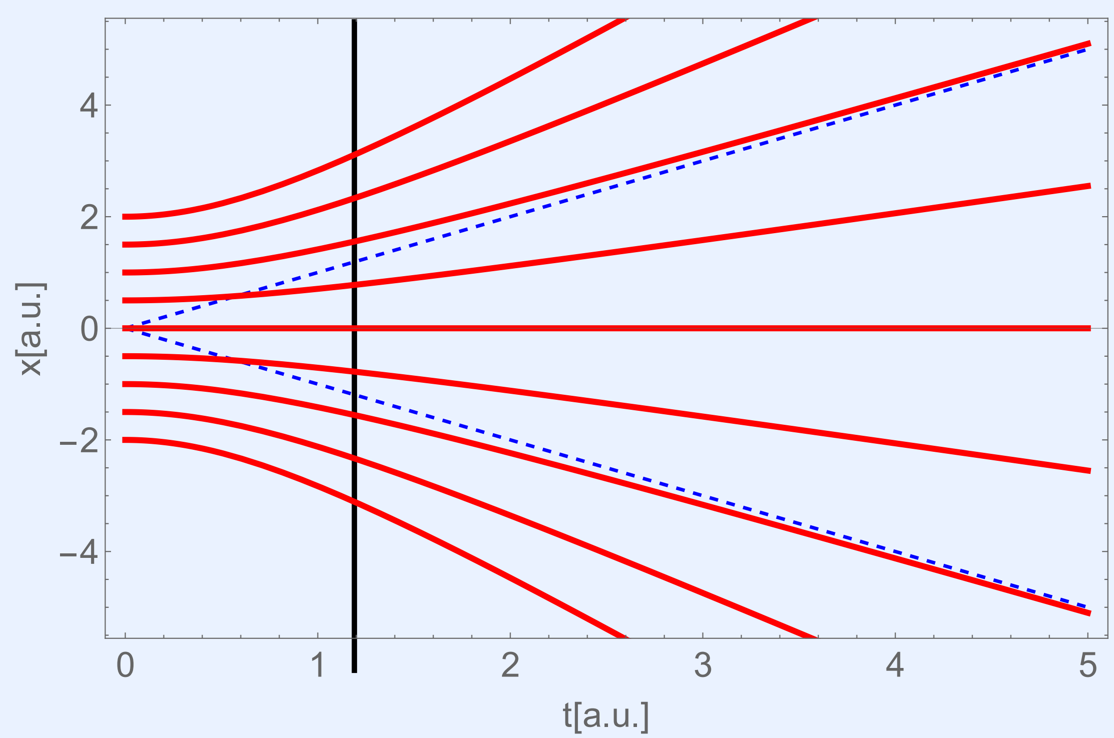
Gaussian beam

Optics



$$\frac{z_R}{c} = \left(\frac{k}{c}\right)\sigma^2$$

QM

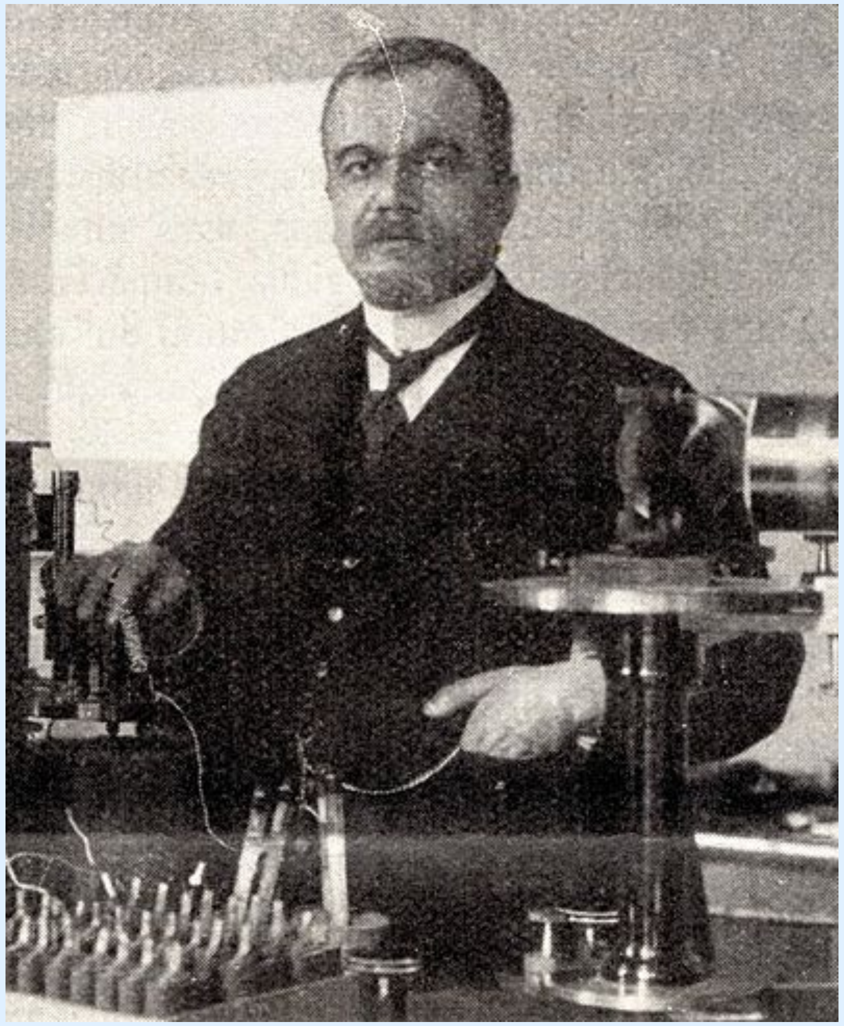


T

$$T = \left(\frac{m}{\hbar}\right)\sigma^2$$

1801

1836



1890

Helmholtz Equation = TISE

The paraxial approximation of Optics

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + 2ik\,\frac{\partial\psi}{\partial z} = 0$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) - i\frac{\hbar^2 k}{m}\,\frac{\partial\psi}{\partial z} = 0$$

Classical z motion

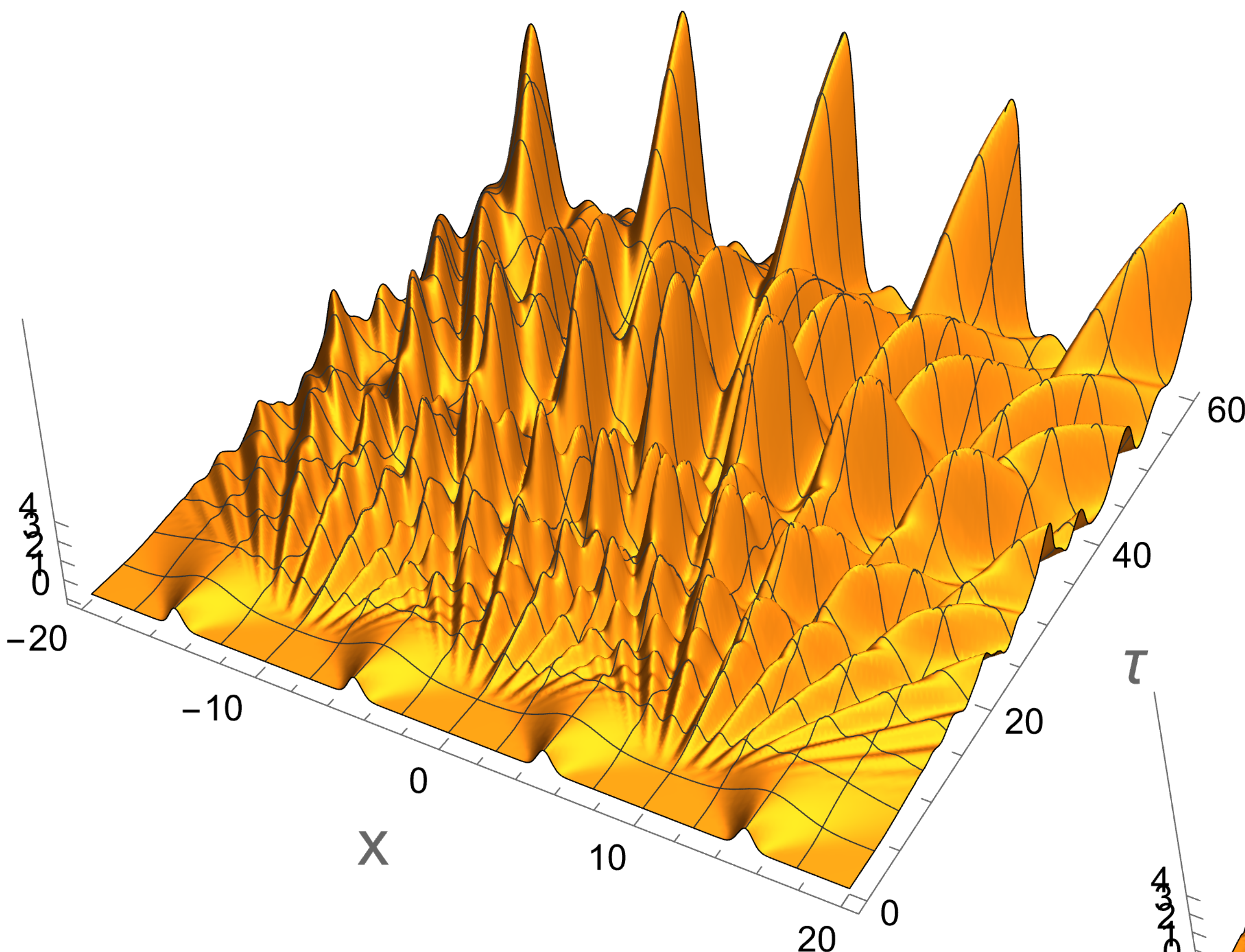
$$\frac{\hbar k}{m} = \frac{p}{m} = \frac{\partial z}{\partial t}$$

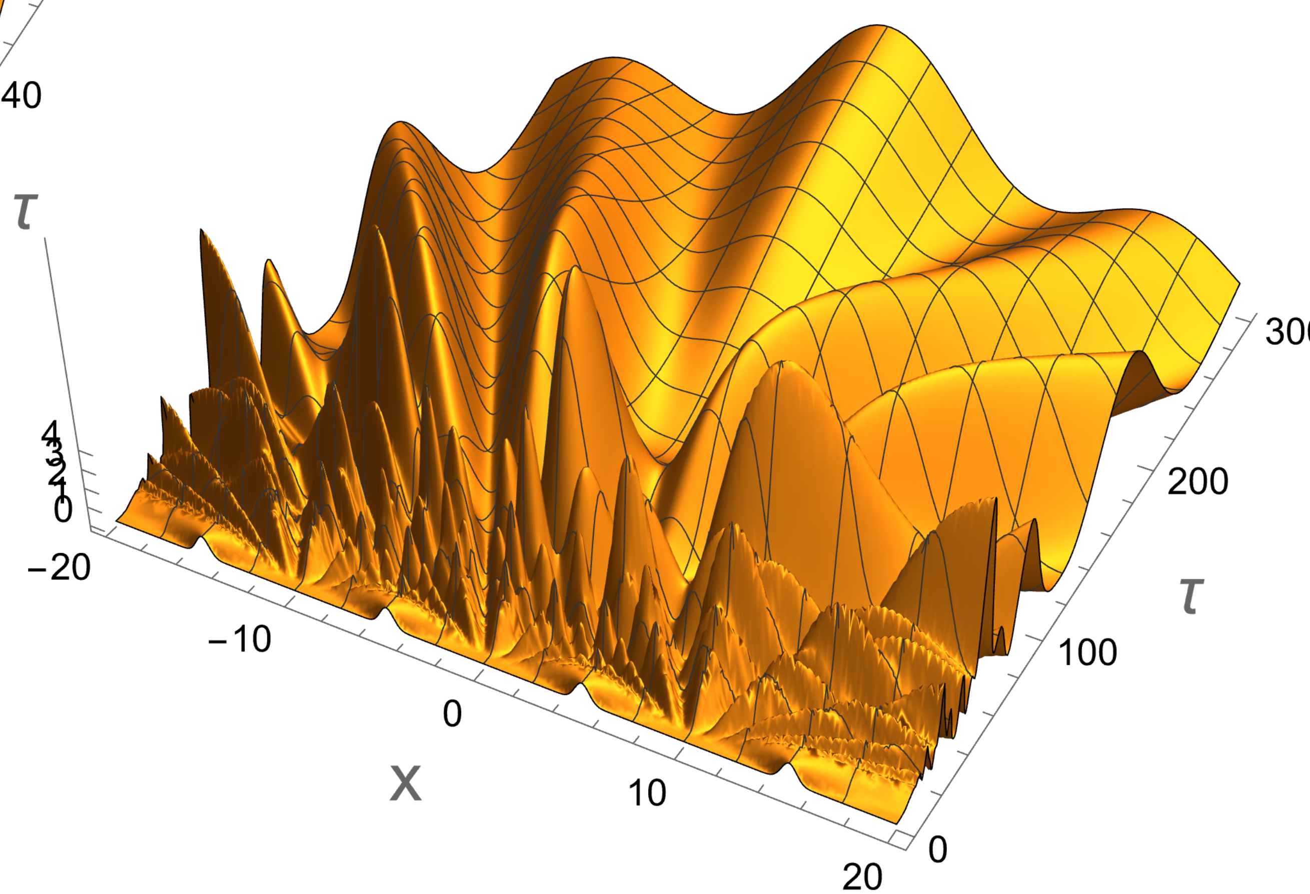
$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) - i\hbar\,\frac{\partial\psi}{\partial t} = 0$$

The TDSE

Lab. frame. Light or particle diffraction through 4 Gaussian slits.

d = 5, sigma = 0.5





Space-Time Transformation of Free Motion

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} - i\hbar\frac{\partial\Psi}{\partial t} = 0.$$

$$\bar{x} \equiv \frac{x}{a(t)} \qquad \bar{t} \equiv \int^t \frac{dt'}{a(t')^2}$$

with the Bohmian choice

$$a(t) = (1+\tau^2)^{1/2} \qquad \tau = t/T$$

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\bar{x}^2} + \frac{1}{2}m\,\omega_0^2\,\bar{x}^2\right)\Phi = i\hbar\frac{\partial\Phi}{\partial\bar{t}}$$

with constant frequency $\omega_0 = 1/T$

in the co-moving frame

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \bar{x}^2} + \frac{1}{2} m\,\omega_0^2\,\bar{x}^2\right)\Phi = i\hbar\,\frac{\partial \Phi}{\partial \bar{t}}$$

has eigenfunctions

$$\Phi_n(\bar{x},\bar{t}) = \frac{1}{(\pi\sigma^2)^{1/4}}\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{\bar{x}}{\sigma}\right)\exp\left(-\frac{\bar{x}^2}{2\sigma^2}\right)\exp\left(-\frac{i}{\hbar}E_n\bar{t}\right)$$

with $\bar{x} = x(t)/a(t) = x_0 = vT$

$$\Phi_n(\bar{x},\bar{\tau}) = \frac{1}{(\pi\sigma^2)^{1/4}}\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{p\sigma}{\hbar}\right) \times \exp\left[-\frac{\sigma^2 p^2}{2\hbar^2}\right]\exp\left(-\frac{i}{\hbar}E_n\bar{t}\right)$$

Along the complete co-moving trajectory the space wave function is proportional to the momentum wave function (Fraunhofer limit)

What is the new time in the energy phase factor ?

$$E_n\bar{t}/\hbar = (n+\frac{1}{2})\bar{t}/T = (n+\frac{1}{2})\arctan\tau$$

$$\bar{\tau} \equiv \bar{t}/T = \arctan\tau = \arctan(t/T)$$

The proper time in the co-moving frame is the Gouy phase !

$$Lab.\ \ time\ \ t/T\ \ from\ \ Zero\ \ to\ \ \infty$$

$$Co-moving\ \ time\ \ \bar{t}/T\ \ from\ \ Zero\ \ to\ \ \pi/2$$

Expand an arbitrary wave packet

$$\chi(\bar{x}, \bar{\tau}\ ) = \sum_n a_n \, \Phi_n(x_0, \bar{\tau}\ ) \qquad \bar{\tau} \equiv \bar{t}/T$$

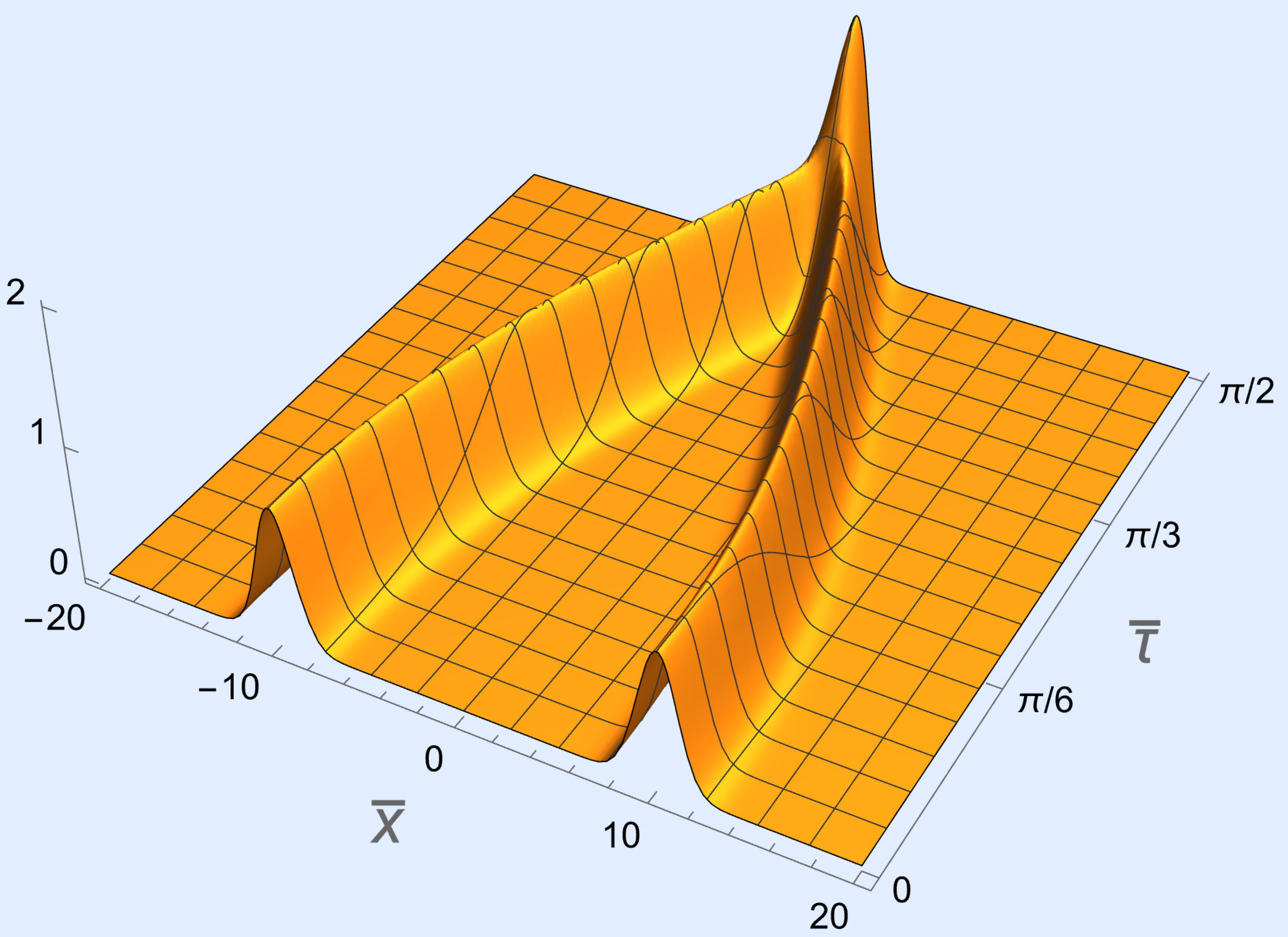
A displaced Gaussian slit function

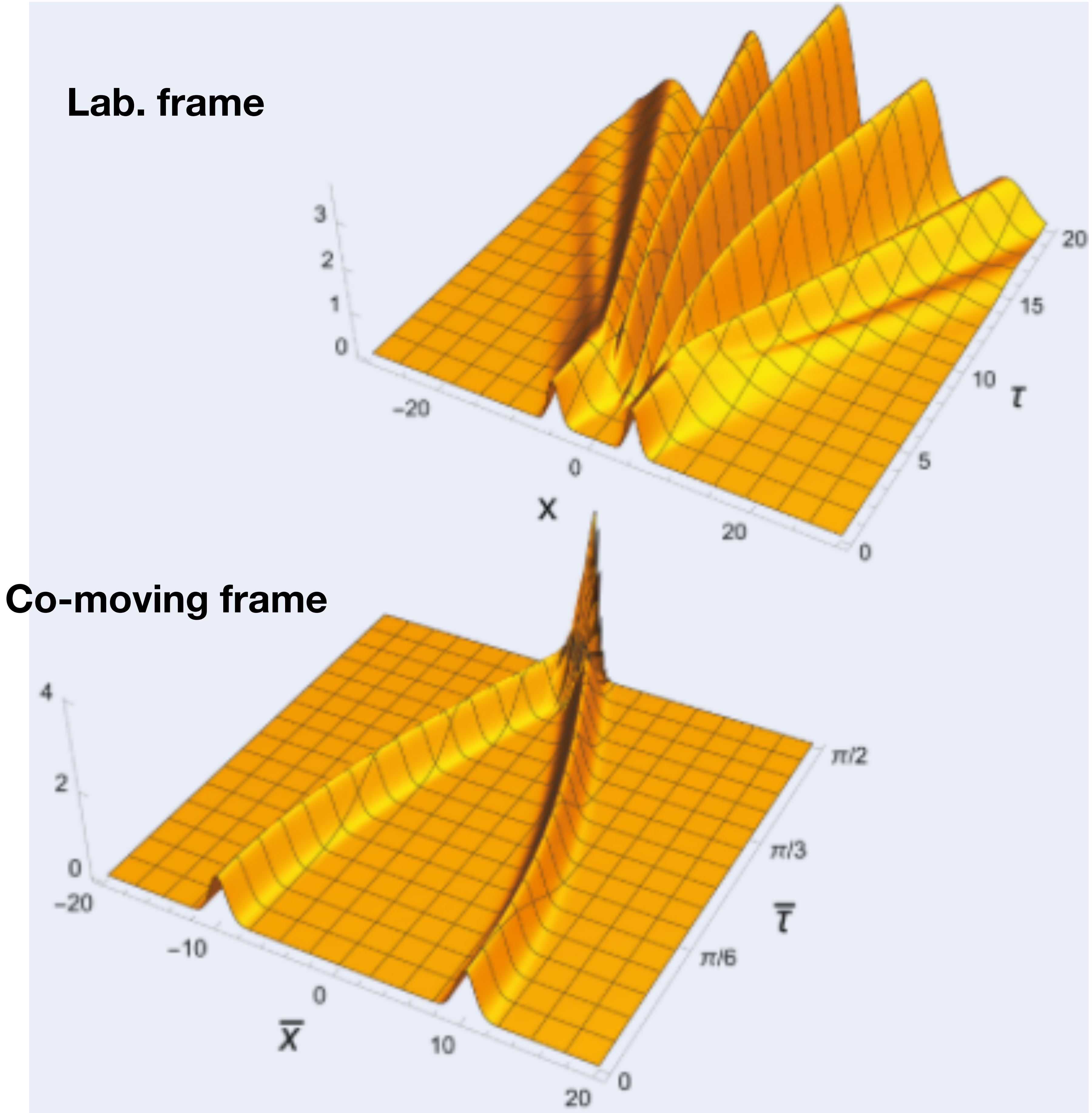
$$\Psi_0(x-d,0) = \frac{1}{(\pi)^{1/4}} \exp\left(-\frac{(x-d)^2}{2}\right)$$

$$|\chi(\bar{x},\bar{\tau})|^2 = \frac{1}{(\pi)^{1/2}} \exp\left(-(x_0 - d\cos\bar{\tau})^2\right)$$

a coherent state

Coherent states, no interference, co-moving frame

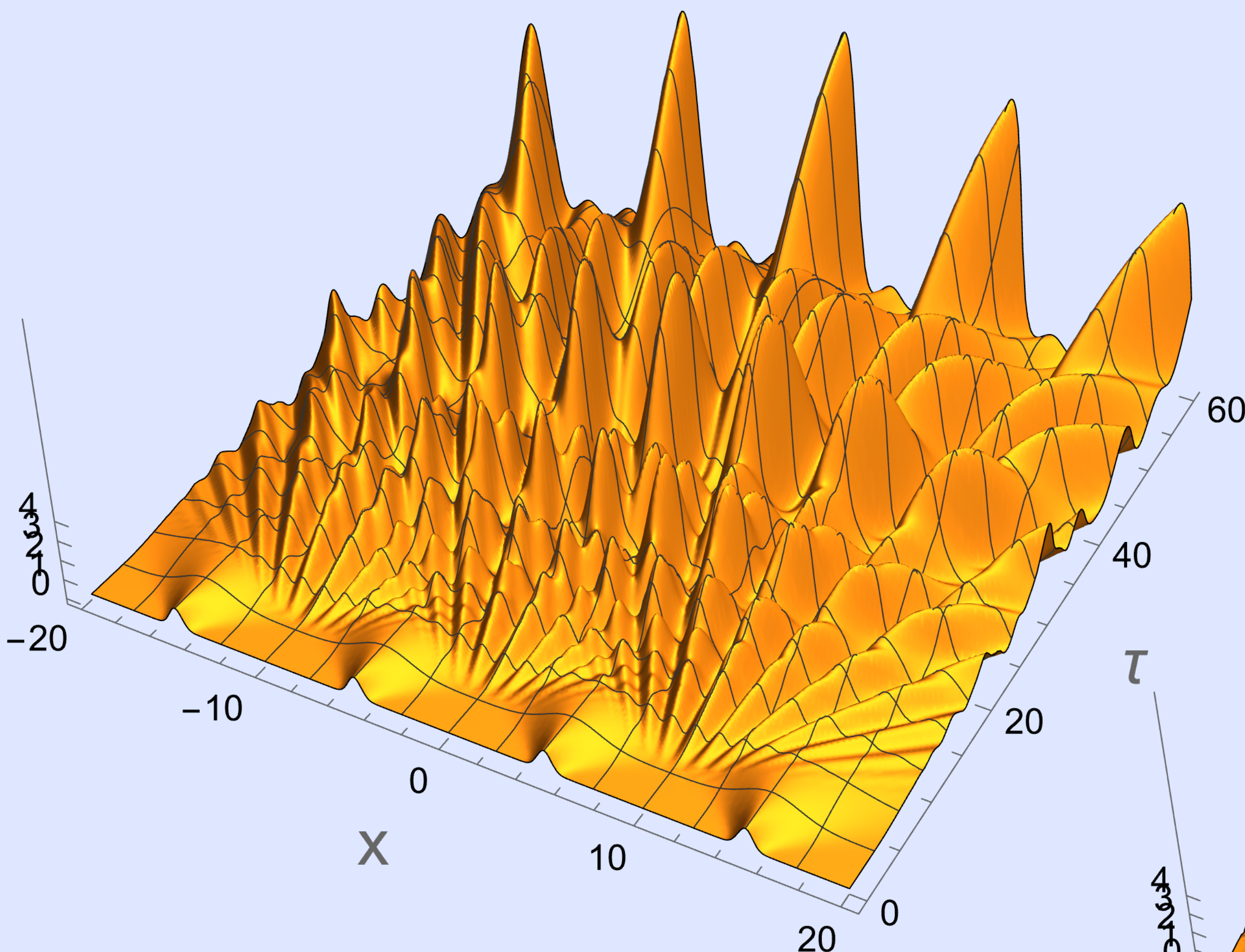


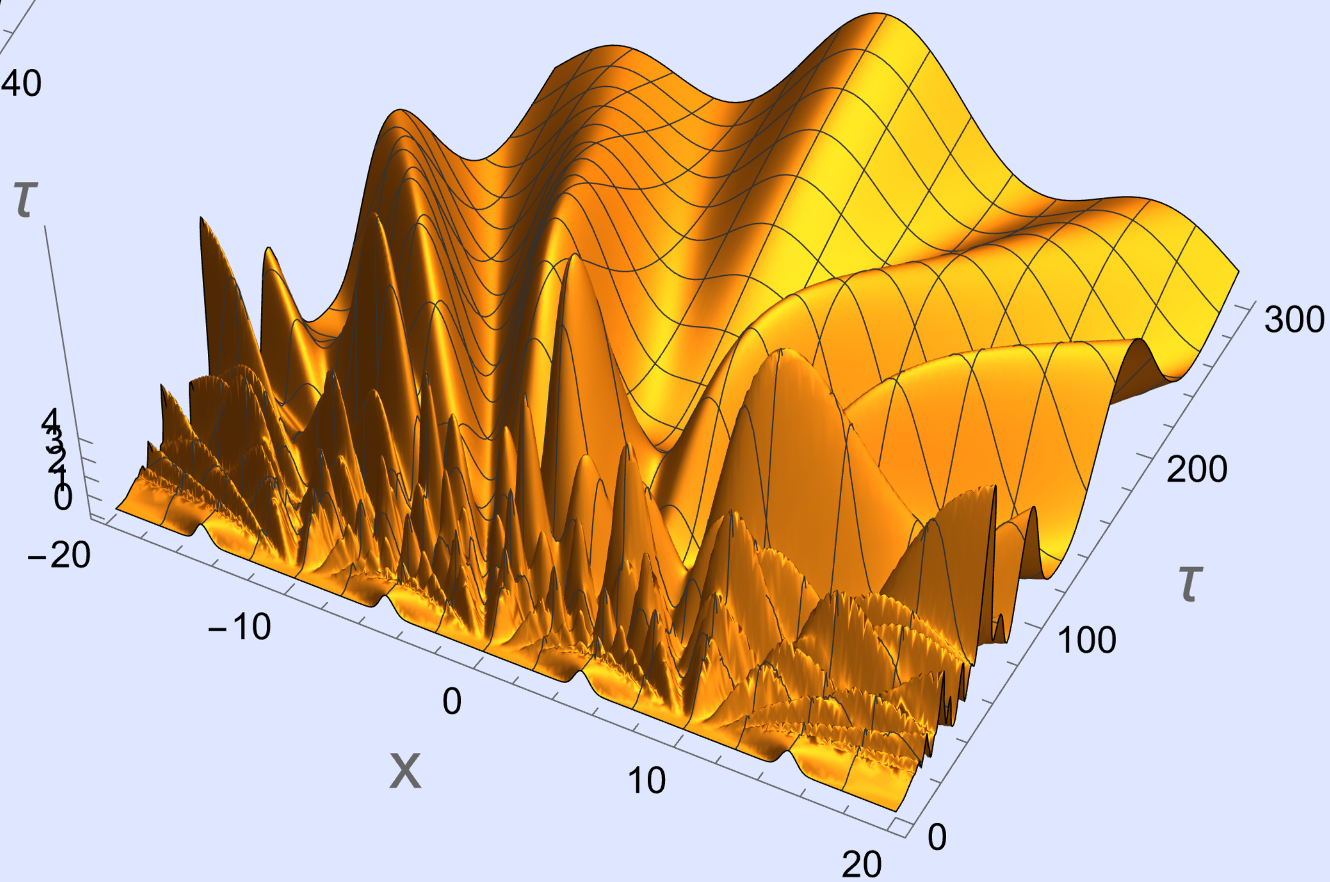


Lab. frame
3
2
1
0
-20
0
X
20
0
5
10
15
20
τ
Co-moving frame
4
2
0
-20
-10
0
X̄
10
20
0
π/6
π/3
π/2
τ̄

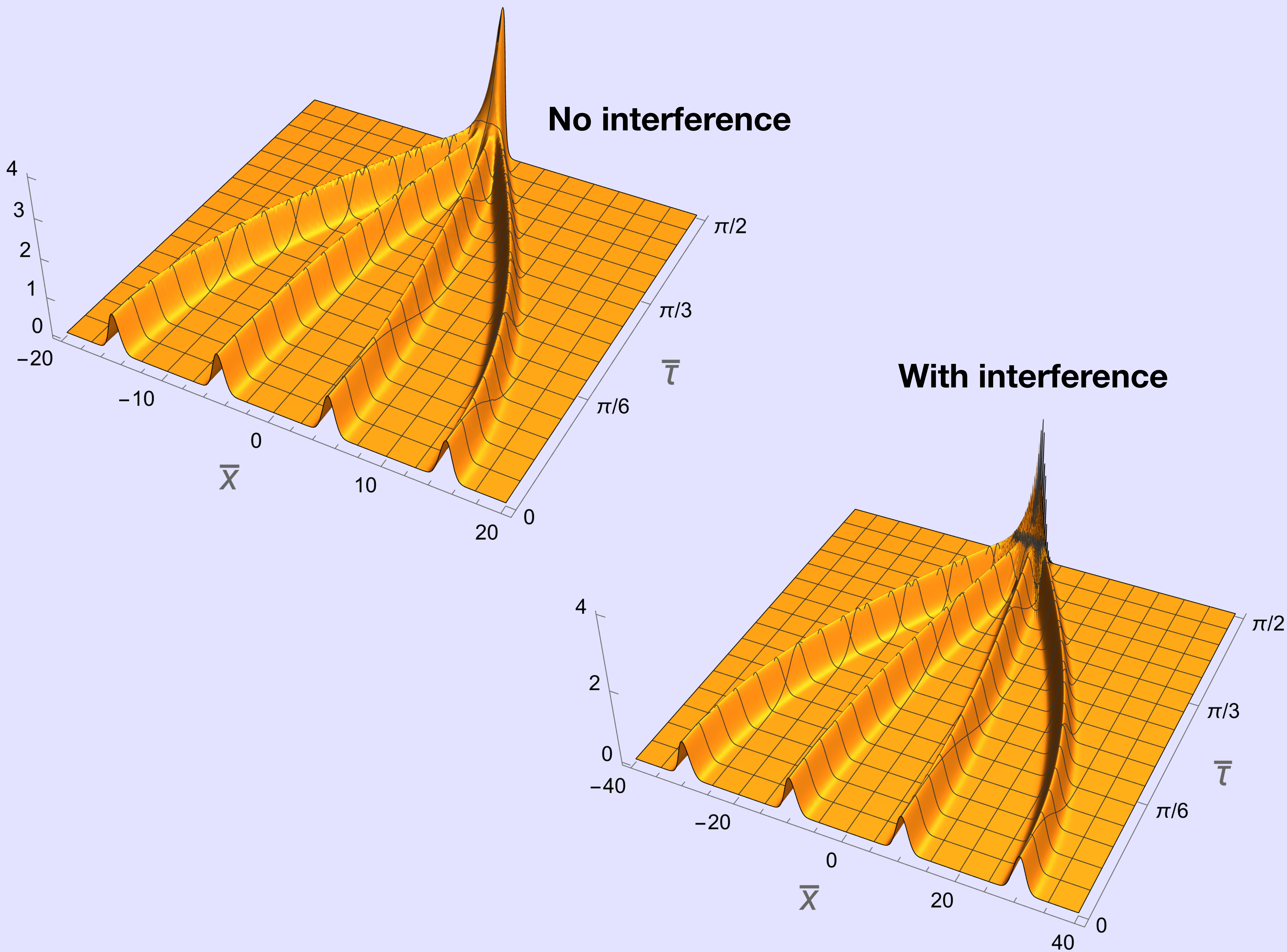
Lab. frame, 4 Gaussian slits

d = 5, sigma = 0.5





Co-moving frame, 4 coherent states



Wave function of the Universe

J. B. Hartle and S. W. Hawking

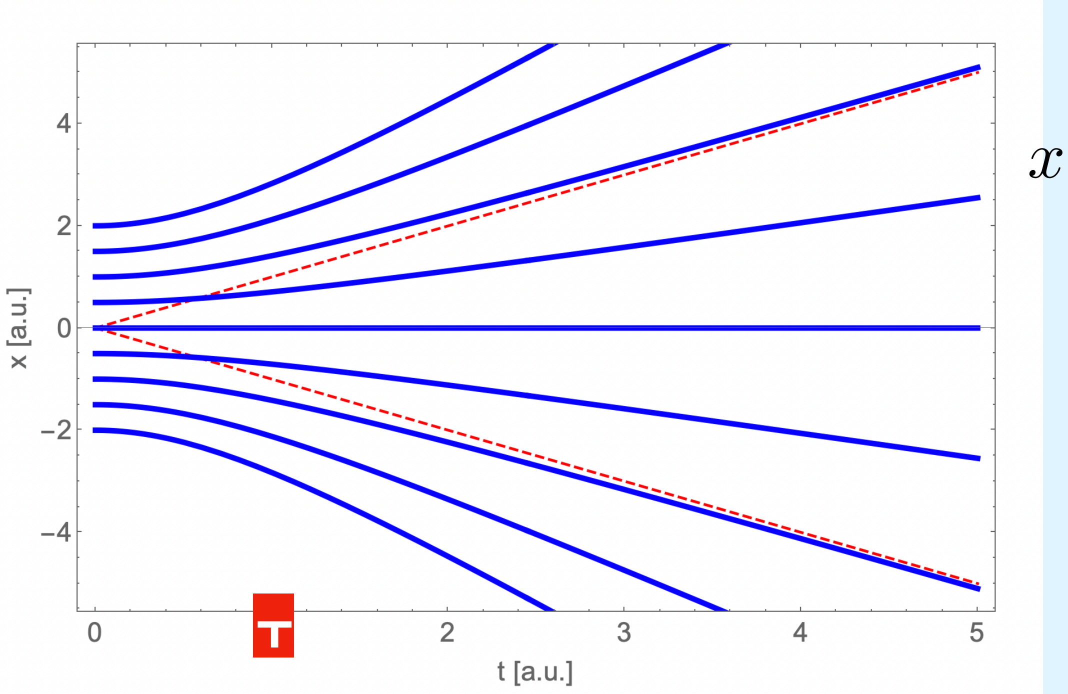
Phys. Rev. D **28**, 2960 – **Published 15 December, 1983**

Hubble "constant"

$$a(t) = (1+\tau^2)^{1/2} = \left(1 + \frac{t^2}{T^2}\right)^{1/2}$$

$$H = \frac{\dot{a}(t)}{a(t)} = \frac{t}{(t^2+T^2)}$$

$$T = \frac{m\sigma^2}{\hbar}$$



$t \gg T$

$x(t) \rightarrow vt$

Natural Sciences 2 (2022): e20210089.

Natural Sciences 4 (2024): e20230012

Natural Sciences 5 (2025): e20240028

European Journal of Physics 45 (2024): 045402.